

Chaos in a generalized Lorenz system

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Abstract

A three-component dynamic system with influence of pumping and nonlinear dissipation describing a quantum cavity electrodynamic device is studied. Different dynamical regimes are investigated in terms of divergent trajectories approaches and fractal statistics. It has been shown, that in such a system stable and unstable dissipative structures type of limit cycles can be formed with variation of pumping and nonlinear dissipation rate. Transitions to chaotic regime and the corresponding chaotic attractor are studied in details.

Key words: Phase space; limit cycle; chaos; fractals.

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1 Introduction

The most intriguing phenomenon in nonlinear dynamical systems theory is a transition from regular dynamics to irregular one. Despite a transition to irregular dynamics can be driven by stochastic sources introduction into corresponding evolution equations, a special interest from the theoretical viewpoint is an observation of such regime in deterministic systems. A famous work by E.Lorenz [1] shows a possibility of regular dynamical systems to exhibit chaotic

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regime. After, a lot of works concerning this problem allowed to develop mathematical tools to study chaotic regime in different kind of physical, chemical, biological and other systems, see Refs.[2,4,3,5], for example.

Mathematically studying of Lorenz system was performed in [6,7], where authors introduced the geometric Lorenz flows. Invariance principle for geometric Lorenz attractors were obtained in [8]. Description of dynamics of the Lorenz system having an invariant algebraic surface presented in [9].

In physics, thermodynamics in particular, chaos is applied in the study of turbulence leading to the understanding of self-organizing systems and system states (equilibrium, near equilibrium, the edge of chaos, and chaos). The occurrence of a chaotic regime is related to the interplay between the instability and nonlinearity. The instability is responsible for the exponential divergence of two nearby trajectories, the nonlinearity bounds trajectories within the finite volume of the phase space of the system. The combination these two mechanisms gives rise to a high sensitivity of the system to the initial conditions.

It is well known, that optoelectronic devices are the simplest systems which manifest chaotic regime under deterministic conditions. In practice, a controlling of dynamical regimes in optoelectronic devices is very actual task, because different types of radiation can be used for different areas of human activity [11,10]. The problem to control chaotic regimes in laser dynamics attracts much more attention in last decades. Nowadays new mechanisms which lead to chaotic regime generation were found [12,13,14]. Moreover, a rapidly developing approach to study the quantum cavity electrodynamic models shows a possibility of chaotic radiation and the chaos control [15,16]. From a laser physics it follows that self-organization processes with dissipative structures formation or generation of chaotic signal can be stimulated by introducing additional nonlinear medium into the Fabry-Perot cavity [21,10,17,20,19,18]. Experimental studies of the chaotic regime in such systems allows to generalize well known theoretical models, identifying additional nonlinearity sources, in order to provide a more detailed theoretical analysis and to predict different types of laser dynamics [23,24,25,22].

From the theoretical viewpoint, a problem to find ability of the system to manifest chaotic regimes and the control of chaos is of a special interest in a nonlinear dynamical system theory. Moreover, due to chaotic dynamics is accompanied by a different type of attractors appearing, another interesting problem is to investigate properties of such attractors. On physical grounds, the related actual problem is to investigate a chaotic regime in a generalized model of a dynamical system, whereas an initial model does not exhibits it itself.

In this paper we solve above problems for a three-parametric dynamical system realized in optoelectronic devices. We are aimed to study a possibility of the well known two-level laser system type of Lorenz-Haken model [11,17], which can not show chaotic regime in actual ranges of system parameters, but manifests it if an additional nonlinear medium is introduced into the cavity. In our investigation we use suppositions of absorptive optical bistability and consider three-parametric system in order to show that a transition to chaotic regime can be controlled by the pumping and a bistability parameter proportional to the atomic density. The last one is related to nonlinear dependence of a relaxation time of one of dynamical variables versus its value. It will be shown that varying in a system parameters one can arrive at stable and unstable dissipative structures, period doubling bifurcation and a chaotic regime. A corresponding chaotic attractor is studied by different approaches.

The paper is organized in the following manner. In Section 2 we present a model of our system where we introduce theoretical constructions to model an influence of a nonlinear dissipation. Section 3 is devoted to development of the analytical approach to study possible regimes. In Section 4 we apply the derived formalism to investigate a reconstruction of the dynamics of the system. Section 5 is devoted to considering the main properties of the chaotic behaviour of a system. Main results and perspectives are collected in the Conclusions.

2 Model

Let us consider a three parametric dynamical model, which is widely used to describe self-organization processes:

$$\begin{cases} \dot{x} = -x + y, \\ \sigma \dot{y} = -y + xz, \\ \varepsilon \dot{z} = (r - z) - xy, \end{cases} \quad (1)$$

where dot means derivative respect to the time t . This model can be simply derived from the well known chaotic Lorenz system [1] using following relations: $t' = \sigma t$, $x = X/\sqrt{b}$, $y = Y/\sqrt{b}$, $z = r - Z$, $\varepsilon = \sigma/b$. Here X , Y , Z are dynamical variables of the Lorenz system; σ , b and r are related constants; we drop the prime in the time t for convenience. The dynamical system (1) can be obtained in the lasers theory where the Maxwell-type equations for electromagnetic field and a density matrix evolution equation are exploited. It leads to the system of the Maxwell-Bloch type that is reduced to the Lorenz-Haken model (1) for two level laser system (see, for example, Refs.[17,10,26]). In such a case x , y , z are addressed to a strength of the electric field, polarization and an inversion population of energy levels. Models type of Eq.(1) are exten-

sively used to describe synergetic transitions in complex systems [27], where above variables x , y , z acquire a physical meaning of an order parameter, a conjugated field and a control parameter; the quantity r relates to the pump intensity and measures influence of the environment.

A naive consideration of dynamical regimes in the model Eq.(1) at $r > 0$ and $\sigma \simeq \varepsilon \simeq 1$ shows that despite the system has nonlinearities in both second and third equations its behaviour is well defined. In this paper we consider a more general case reduced to the absorptive optical bistability models [20,17]. From the physical viewpoint absorptive optical bistability is related to possibility of the additional medium in the Fabry-Perot cavity (phthalocyanine fluid [18], gases SF_6 , $BaCl_3$ and CO_2 [19,17]) to transit a radiation with high intensities only with absorption of weak signals. Despite models of such a type were considered in the one-parametric case and only a stationary picture of dynamical system behaviour is analyzed (see Ref.[28] and citations therein), we will focus onto describing possible dynamical regimes in the three-parametric model. Mathematically, our model reads

$$\begin{cases} \dot{x} = -x + y + f_\kappa, & f_\kappa = -\frac{\kappa x}{1 + x^2}, \\ \sigma \dot{y} = -y + xz, \\ \varepsilon \dot{z} = (r - z) - xy. \end{cases} \quad (2)$$

Here the additional force f_κ is introduced to take into account influence of an additional medium; κ is the so-called bistability parameter proportional to the atomic density. Formally, it means a dispersion of a relaxation time for the variable x . Indeed, if we rewrite the first equation in the form $\dot{x} = -x/\tau(x) + y$, then the dispersive relaxation time is $\tau(x) = 1 - \kappa/(1 + \kappa + x^2)$.

Our main goal is to study possible scenarios of a dynamical regimes reconstruction in such three-parametric system under influence of the pumping r and the nonlinear dissipation controlled by κ . We will investigate conditions of a chaotic regime appearance. The corresponding irregular dynamics will be studied in details. It will be shown that a chaotic regime can be controlled by pumping and properties of an additional medium.

3 Stability analysis

Let us consider conditions where all variables are commensurable, setting $\sigma, \varepsilon \sim 1$. A behaviour of the system we present in the phase space (x, y, z) . At first, let us study steady states. Setting $\dot{x} = 0$, $\dot{y} = 0$ and $\dot{z} = 0$, one can find fixed points described by the values $x_0^{(k)}$, $y_0^{(k)}$ and $z_0^{(k)}$, where $k = 1, 2, 3$. The

first one has coordinates

$$(x_0^{(1)}, y_0^{(1)}, z_0^{(1)}) = (0, 0, r) \quad (3)$$

and exists always; two symmetrical points, denoted with superscripts 2 and 3 respectively, have coordinates

$$\begin{aligned} (x_0^{(2,3)}, y_0^{(2,3)}, z_0^{(2,3)}) = \\ \left(\pm \sqrt{r - \kappa - 1}, \frac{\pm r \sqrt{r - \kappa - 1}}{r - \kappa}, \frac{r}{r - \kappa} \right) \end{aligned} \quad (4)$$

and realized under the condition

$$r \geq \kappa + 1. \quad (5)$$

It means, that $r_c = \kappa + 1$ is a critical point for the bifurcation of doubling.

According to the stationary states behaviour under influence of pumping and nonlinear dissipation, let us perform a local linear stability analysis [29,30]. Within the framework of the standard Lyapunov exponents approach time depended solutions of the system (2) are assumed in the form $\vec{u} \propto e^{\Lambda t}$, $\Lambda = \lambda + i\omega$, where $\vec{u} \equiv (x, y, z)$, λ controls the stability of the stationary points, ω determines a frequency, as usual. Magnitudes for real and imaginary parts of Λ are calculated according to the Jacobi matrix elements $M_{ij} \equiv \left(\partial f^{(i)} / \partial u_j \right)_{u_j = u_{j0}}$, $i, j = x, y, z$; $f^{(i)}$ represents the right hand side of the corresponding dynamical equations in Eq.(2); the subscript 0 relates to the stationary value. As a result, the Jacobi matrix takes the form

$$M = - \begin{pmatrix} 1 + \kappa \frac{1 - x_0^2}{(1 + x_0^2)^2} + \Lambda & -1 & 0 \\ -z_0 & 1 + \Lambda & -x_0 \\ y_0 & x_0 & 1 + \Lambda \end{pmatrix}. \quad (6)$$

Let us define the stability of the fixed point (3). A solution of the eigenvalue problem yields that the three eigenvalues are real and negative:

$$\begin{aligned} \Lambda_1 = \lambda_1 &= -1 < 0, \\ \Lambda_2 = \lambda_2 &= -\frac{1}{2}\kappa - 1 - \frac{1}{2}\sqrt{\kappa^2 + 4r} < 0, \\ \Lambda_3 = \lambda_3 &= -\frac{1}{2}\kappa - 1 + \frac{1}{2}\sqrt{\kappa^2 + 4r} < 0. \end{aligned} \quad (7)$$

It means that if both κ and r satisfy the condition (5), then the phase space is characterized by only node (3).

On the other hand, at $r > r_c$, all three fixed points are realized in the phase space. As it follows from our analysis the fixed point (3) changes its stability and becomes a saddle, due to $\lambda_3 > 0$.

The stability of other two points with coordinates given by Eq.(4) can be set from solutions of the cubic equation

$$\begin{aligned} \Lambda^3 + \left[3 + \kappa \frac{2 + \kappa - r}{(r - \kappa)^2} \right] \Lambda^2 \\ + \left[2 + \kappa - r - \frac{r}{r - \kappa} + 2\kappa \frac{2 + \kappa - r}{(r - \kappa)^2} \right] \Lambda \\ + \left[2 + \kappa + \kappa \left(\frac{2 + \kappa - r}{r - \kappa} \right)^2 \right] = 0. \end{aligned} \quad (8)$$

Analytically, we can define only a border of the domain of system parameters where periodic solutions are realized. To consider a possibility of limit cycles formation in the phase space let us assume $\Lambda = \pm i\omega$ and insert it into Eq(8). It yields the equation for such a border in the form

$$\begin{aligned} 2\kappa \frac{2 + \kappa - r}{(r - \kappa)^2} \left[\kappa - r - 6 + \frac{r}{2} \frac{1}{r - \kappa} \right] \\ - 2\kappa^2 \left(\frac{2 + \kappa - r}{(r - \kappa)^2} \right)^2 \\ + 4(\kappa - 1) + 3r \left(\frac{1}{r - \kappa} - 1 \right) = 0. \end{aligned} \quad (9)$$

As it will be seen below inside this domain in the plane (r, κ) only stable periodic solutions are realized.

4 Analysis of dynamical regimes

To describe the dynamical regimes of the system under consideration we calculate the phase diagram shown in the plane (r, κ) in Fig.1. Here the dashed line $r = \kappa + 1$ divides the plane (r, κ) into two domains denoted as I and II and relates to the critical values of r and κ . Inside the domain I only node is realized. The corresponding phase trajectories are shown in Fig.2a. Above the dashed line a set of different dynamical regimes can be observed. Inside the domain II the node point transforms into a saddle point and two additional stable focuses appear (see Fig.2b), the corresponding Lyapunov exponents has negative real parts, and equivalent imaginary ones. In the domain bounded by the solid line stable limit cycles are formed. This line is obtained as a solution of Eq.(9) with assumption that dynamical solutions of the system Eq.(2) are

periodic, i.e. $\Lambda = \pm i\omega$. When we move from the domain II to the domain III above focuses loss their stability and, as result, stable limit cycles are realized. All phase trajectories are attracted by these manifolds. The corresponding limit cycles with unstable focuses inside of them are shown in Fig.2c. The domain IV bounded by solid, dotted and dash-dotted lines corresponds to the case where two stable and the corresponding two unstable limit cycles appear (see Fig.2d). Such a transformation of dynamical system behaviour (transition from domain III into domain IV) is a result of double Hopf's bifurcation. Here unstable focuses, shown in Fig.2c, loss their unstability that leads to appearing unstable limit cycles inside stable ones (see Fig.2d). When we cross the dotted line (transition from the domain II into domain IV) the stability of focuses shown in Fig.2b is not changed, but two stable limit cycles and two unstable limit cycles appear, as Fig.2d shows. A scenario of such a bifurcation is as follows. With an increase in one of two system parameters r or κ in a point from the dotted line a new manifold type of semi-stable limit cycle is formed [2]. When we move to the domain IV this semi-stable limit cycle decomposes into the both outer stable and inner unstable limit cycles. Moving in opposite direction ($IV \rightarrow II$), one of the well known type of Hopf bifurcation occurs: an annihilation of both stable and unstable limit cycles.

Let us consider a change of the system behaviour when we move from the domain III to the domain V bounded by solid and dash-dotted lines. If the dash-dotted line is crossed, then doubling period bifurcation occurs. Here two stable limit cycles with unstable focuses, shown in Fig.2c, are transformed into the one limit cycle (Fig.2e), whose period equals two periods of the one cycle, shown in Fig.2c. The behaviour of the system inside the domain V is the same as shown in Fig.2e. If we move from the domain V to the domain VI we get the period doubling scenario of chaos formation (see Fig.2f). If the line that divides both domains V and VI is crossed, then the stability of focuses shown in Fig.2e is changed. It leads to a formation of two unstable

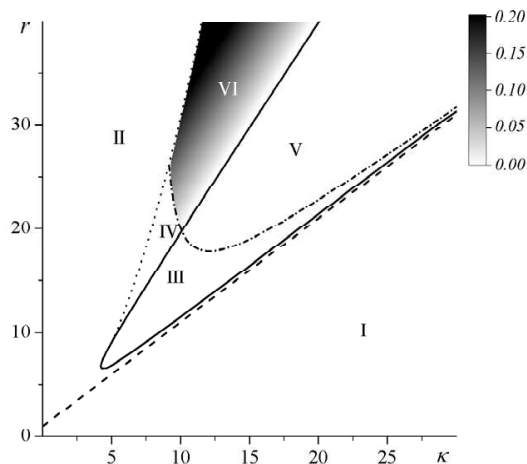


Fig. 1. Phase diagram of the system (2)

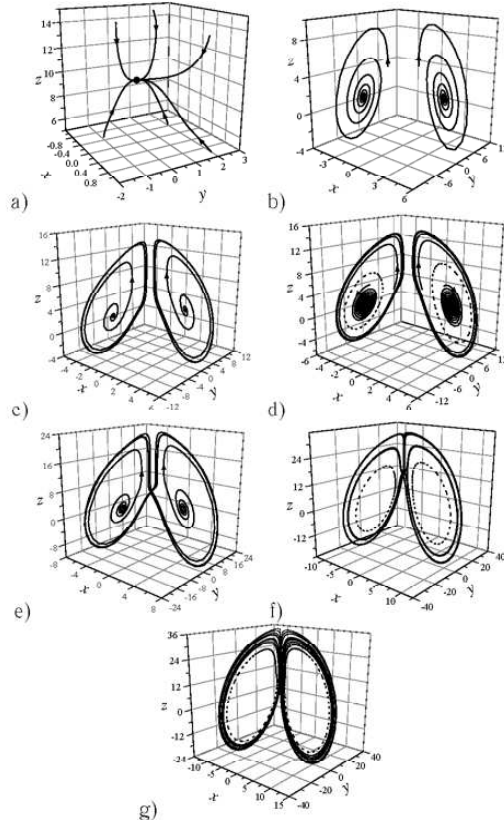


Fig. 2. Phase portraits of the system (2) at : a) $r = 10.0$, $\kappa = 15.0$; b) $r = 15.0$, $\kappa = 5.0$; c) $r = 15.0$, $\kappa = 10.0$; d) $r = 20.0$, $\kappa = 9.0$; e) $r = 25.0$, $\kappa = 15.0$; f) $r = 35.0$, $\kappa = 12.0$; g) $r = 39.0$, $\kappa = 11.5$

limit cycles and the additional doubling period bifurcation. If we plunge into the dark zone the phase trajectories become irregular (see Fig.2g) due to the period doubling bifurcations. In such a case we obtain the chaotic attractor bounded by outside surface. Properties of such an attractor will be studied below. If we move from the domain IV to the domain VI the standard doubling period bifurcation occurs (see transition from Fig.2d to Fig.2f). The transition from the domain II into the domain VI is described by the following scenario. In the line that separates both domains II and VI a complex situation is realized. Here transverse intersections of both stable and unstable manifolds of the hyperbolic stationary point (saddle) occur. This indicates a homoclinic structure and stochastic layer appearance. In such a case the phase space is characterized by two unstable limit cycles and irregular behavior of trajectories outside of them (see Fig.2g).

To elucidate that we deal with irregular behaviour let us use a quantity h

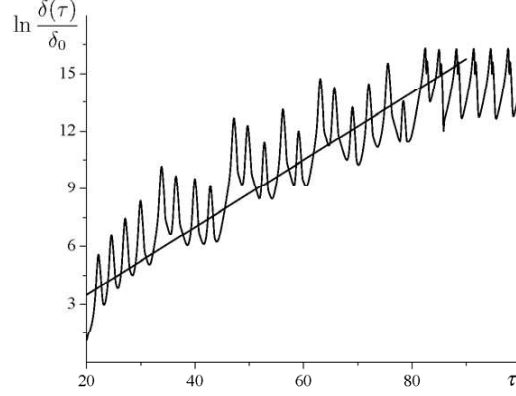


Fig. 3. The time evolution of a distance $\delta(\tau)$ between two nearest trajectories in the attractor Fig.2g at $\delta(\tau_0) = 10^{-5}\sqrt{3}$

known as Kolmogorov-Sinai entropy defined as follows

$$h = \lim_{\substack{\delta(\tau_0) \rightarrow 0 \\ \tau \rightarrow \infty}} \frac{1}{\tau} \ln \left[\frac{\delta(\tau)}{\delta(\tau_0)} \right].$$

This quantity allows to observe behaviour of two trajectories 1 and 2 originating at $t = \tau_0$ from two starting points $\vec{u}_1(\tau_0)$ and $\vec{u}_2(\tau_0)$. It is well known that if $h > 0$ then the system manifests chaotic motion. The corresponding calculations of the separation $\delta(\tau) = |\vec{u}_2(\tau) - \vec{u}_1(\tau)|$ between these trajectories versus time τ when the trajectories are in attractor is shown in Fig.3. Defining the average inclination angle for the fluctuating curve, one can define Kolmogorov-Sinai entropy $h = 0.1749$. According to the definition of h the characteristic mixing time for the system $t_{mix} \equiv h^{-1} = 5.7175$ in the time units. At $t \ll t_{mix}$ the behaviour of a system is well predicted, whereas at $t \gg t_{mix}$ only probabilistic description of the system dynamics is justified. From Fig.3 it follows that our attractor has a bounded size due to that distance $\delta(\tau)$ saturates at large time τ .

Despite the value of h might also depend on the choice of initial points of the trajectories it gives not only the qualitative, but also a quantitative specification of the dynamical regime. The more general criterion which takes into account the direction of the vector of the initial shift $\vec{\delta}(\tau_0)$ is the maximal (global) Lyapunov exponent defined in the form

$$\Lambda_m \equiv \Lambda(\vec{\delta}(\tau_0)) = \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \left\| \frac{\delta \vec{u}(\tau)}{\delta \vec{u}(\tau_0)} \right\|$$

where an upper limit and the norm $\|\vec{u}\|$ are used. The quantity Λ_m allows to consider a divergence of any two initially close trajectories in phase space starting from points $\vec{u}(\tau)$ and $\vec{u}(\tau')$. It follows, that the divergence of such trajectories is given by the dependence $\delta \vec{u}(\tau) = \delta \vec{u}(\tau_0) e^{\Lambda_m \tau}$. It is seen, that

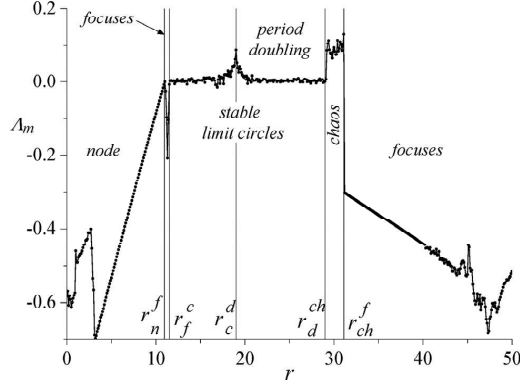


Fig. 4. Dependence of the maximal Lyapunov exponent vs. pumping r at $\kappa = 10.0$

at $\Lambda_m < 0$ trajectories of dynamical system converge and hence we get the regular dynamics, where the trajectories moves toward fixed points in the phase space. In the limit case $\Lambda_m = 0$ one gets the limit cycle in the phase space. If $\Lambda_m > 0$ trajectories diverge and the system shows the chaotic regime. To determine the maximal Lyapunov exponent we use G.Benettin's algorithm [31]. The corresponding spectrum of maximal Lyapunov exponents Λ_m and the domain of its magnitudes are shown in Fig.1.

To prove that above irregular behaviour of phase trajectories can be understood as chaos let us describe in details a behaviour of the maximal Lyapunov exponent Λ_m versus the pump intensity r at fixed κ . The corresponding graph is shown in Fig.4. It is seen that in the domain of the pump intensity $r \in [0, r_n^f]$ one gets $\Lambda_m < 0$. It means that in the phase space only node point is realized. If one chooses $r \in (r_n^f, r_f^c]$, then two stable focuses divided by the saddle point are realized with $\Lambda_m < 0$. In the domain $r \in (r_f^c, r_d^{ch}]$ stable focuses transform into stable limit circles. One needs to note that at $r_c^d < r \leq r_d^{ch}$ the doubling period bifurcation takes place. At the such bifurcation point the theoretical value for the maximal Lyapunov exponent is $\Lambda_m = 0$. However, in computer simulations $\Lambda_m \neq 0$, here a strong nonlinearity in the vicinity of the saddle point acts crucially and hence the obtained values $\Lambda_m > 0$ should be understood as an artifact of computer simulations ¹. The critical value r_c^d of the period doubling bifurcation in our system is obtained when the maximal Lyapunov exponent takes marginal value inside the above domain. In the interval $r \in (r_d^{ch}, r_{ch}^f]$ we get $\Lambda_m > 0$ that corresponds to divergence of phase trajectories and leads to chaotic regime formation. At $r > r_{ch}^f$ the phase space is characterized by stable focuses.

¹ To prove this we compare a similar doubling period bifurcation in Roessler system and have found that Λ_m abruptly changes it values in the vicinity of such bifurcation point.

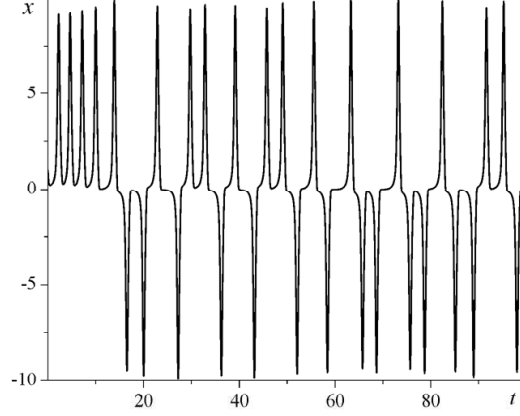


Fig. 5. Time dependence $x(t)$ at $r = 39.0$, $\kappa = 11.5$ from Fig.2g

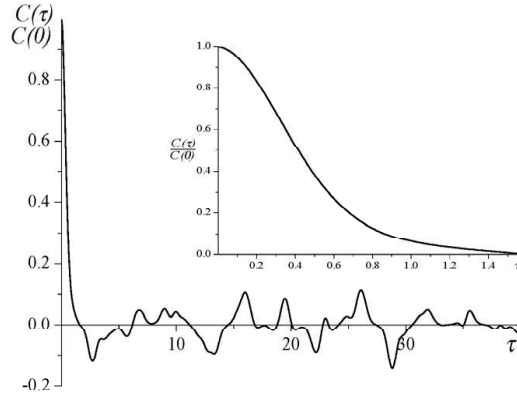


Fig. 6. Auto-correlation function of process $x(t)$ at $r = 39.0$, $\kappa = 11.5$. The corresponding dependence at small time is presented in the insertion

5 Properties of the chaotic regime

Let us consider statistical properties of irregular behaviour of the system, which corresponds to the phase portrait in Fig.2g in details. It is well known that a chaotic regime is characterized by fast decay of an auto-correlation function and continuous character of frequency spectrum. To show that our attractor acquires such properties let us calculate an auto-correlation function $C(\tau) = (2/T) \int_0^{T/2} x(t)x(t+\tau)dt$ of the process $x(t)$ shown in Fig.5. The auto-correlation function is presented in Fig.6. It is seen, that auto-correlation function $C(\tau)$ has fast falling down character as in chaotic systems. It means that the system losses its memory very quickly. Another criterion to show the chaotic regime is a continuous frequency spectrum. To obtain it let us use the Fourier transformation of the correlation function that gives power spectrum function $S(\omega) = \sqrt{2/\pi} \int_0^T C(\tau)e^{i\omega\tau}d\tau$. The corresponding frequency spectrum is shown in Fig.7. It is seen, that spectral density function has a continuous character. Such a picture is typical for chaotic regimes [4].

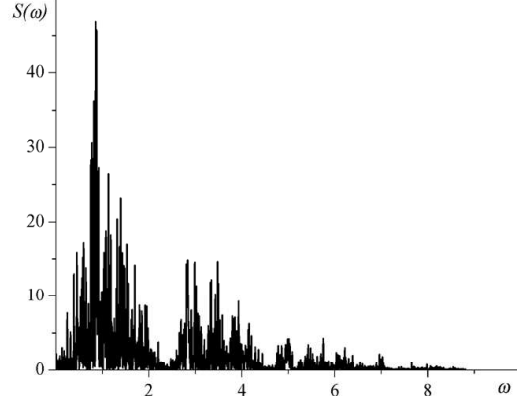


Fig. 7. Spectral density $S(\omega)$ for time series in Fig.5

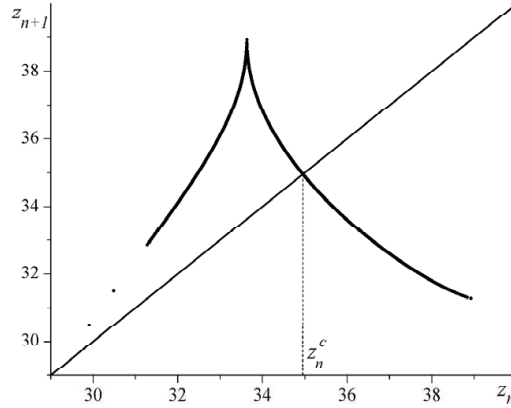


Fig. 8. The one-dimensional law $z_{n+1} = f(z_n)$ for the system (2) at $r = 39.0$, $\kappa = 11.5$

Finally, considering chaotic systems it is interesting to find a one-dimensional law that allows to represent genuine chaotic behaviour. To this end we calculate the first return map $z_{n+1} = f(z_n)$ by fixing the consecutive values of the maxima of the function $z(t)$. Mathematically these maximal values are z -coordinates of points in the Poincare cross-section of the surface $(r-z)-xy = 0$ ($\dot{z} = 0$ due to the third equation in the Eq.(2)). The corresponding one-dimensional law $z_{n+1} = f(z_n)$ is shown in Fig.8.

It is principally that points (z_n, z_{n+1}) lie on the one-dimensional curve with an acute peak with a good accuracy. From the figure Fig.8 it follows that if the initial value $z_1 < z_n^c$, then the chaotic character of dependence $z_n(n)$ is observed. The corresponding dependence z_n versus number of returns is shown in Fig.9a, where chaotic character is visually seen. In the opposite case, when $z_1 \geq z_n^c$ trajectories will converge to the fixed point z_n^c as Fig.9b shows.

Other important physical quantity characterizing classical dynamic stochasticity is the fractal dimension of the attractor. For defining of fractal dimension we will use algorithm of Grassberger-Procaccia [32]. Let we have set of state

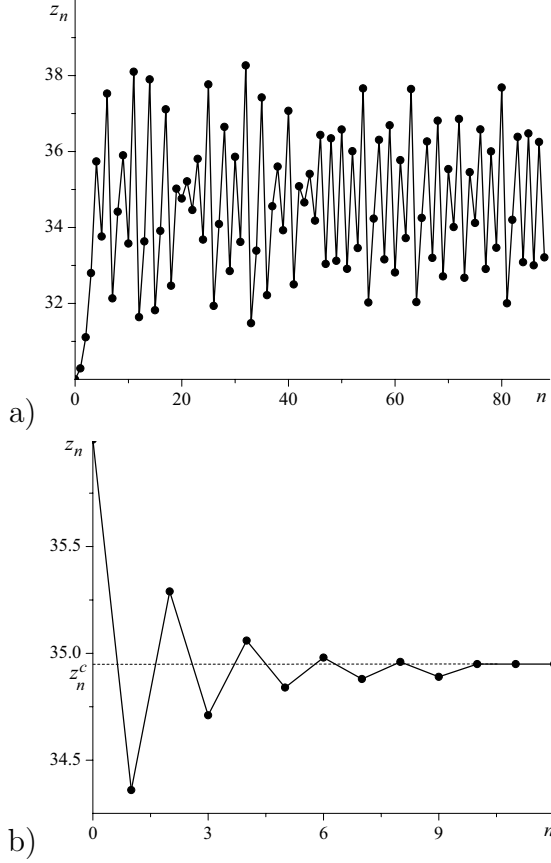


Fig. 9. Dependence $z_n(n)$ for the different initial values z_1 with $z_n^c = 34.95$ at $r = 39.0$, $\kappa = 11.5$: (a) $z_1 < z_n^c$, $z_1 = 30.0$; (b) $z_1 > z_n^c$, $z_1 = 36.0$

vectors \vec{u}_i , $i = 1, 2, \dots, N$ corresponding to the successive steps of numeric integration. In our case \vec{u}_i is the complete set of variables (2) with the values corresponding to the moments of time $t = t_i$. Then we can use numeric data for estimation of the following expression

$$C(\varepsilon) = \lim_{N \rightarrow \infty} \frac{1}{N(N-1)} \sum_{i,j=1}^N \theta(\varepsilon - \|\vec{u}_i - \vec{u}_j\|)$$

where $\theta(\varphi)$ is a step function. According to the Grassberger-Procaccia method, fractal (correlation) dimension of the attractor may be defined as

$$D_c = \lim_{\varepsilon \rightarrow 0} \frac{\log C(\varepsilon)}{\log(\varepsilon)}.$$

An expected dependence of $C(\varepsilon)$ is ε^{D_c} . So, the corresponding plot in double logarithmic scale must be a line with angular coefficient D_c . Results of numeric calculations are presented in Fig.10. A solid line corresponds to least-squares approximation of the results of data processing with $D_c \approx 1.84 \pm 0.01$. Following Ruelle and Takens [33] one can say that our attractor can be identified as a strange chaotic attractor.

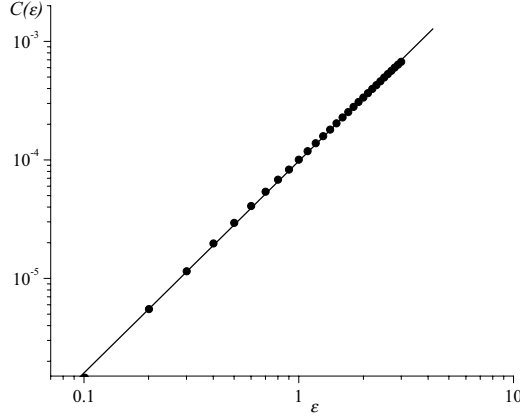


Fig. 10. Dependence of $C(\varepsilon)$ on the values of ε , plotted by numerical integration of the set of equations (2) at $r = 39.0$, $\kappa = 11.5$

6 Conclusions

We have studied different types of dynamical regimes for the three-component system with influence of pumping and nonlinear dissipation. It was shown, that system under consideration is related to absorptive optical bistability systems. Using Lyapunov exponents approach we have shown, that varying in system parameters different types of stable dynamic behaviour of the system are realized. It was found, that controlling the pumping and/or properties of additional nonlinear force, related to absorptive medium in optical systems, unstable and different types of stable dissipative structures can be formed. Considering dynamics in the three-dimensional phase space a set of bifurcations from the limit cycle is studied in details.

It was shown, that in the system under consideration a strange chaotic attractor is realized only if additional nonlinear medium is introduced into the optical cavity. Properties of such a chaotic attractor were investigated using Kolmogorov-Sinai entropy, maximal Lyapunov exponents map, spectrum of the attractor, one-dimensional first return map, and fractal analysis. It was shown, that two neighbour trajectories in the attractor are divergent with positive Kolmogorov-Sinai entropy. The corresponding frequency spectrum is continuous, the temporal auto-correlation function decays fast to zero. From the first return map it follows, that there is a unique initial point exist that separates regular and chaotic behaviour of the system. At last, a fractal (correlation) dimension of the attractor is $D_c \approx 1.84 \pm 0.01$. From above criteria on can conclude that obtained attractive manifold can be considered as a strange attractor as follows from Ruelle and Takens work [33]. Obtained chaotic regime can be controlled by nonlinear dissipation rate and pumping. Varying such parameters we can obtain a stable focus and stable limit cycle from the chaotic regime.

Our results are in good correspondence with well known results in systems with absorptive optical bistability and related to real experimental data for lasers of “C”-class [10,21]. It can be applied to study chaotic regimes in different kinds of dynamical system that can represent self-organization processes in models self-consistently described by hydrodynamic amplitude, conjugated field and control parameter in the form presented by the Lorenz-Haken model.

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